Synopsis

In a foundational paper "Operators Possesing an Open Set of Eigenvalues" written several decades ago, Cowen and Douglas showed that an operator T on a Hilbert space $\mathcal{H}$ possessing an open set $\Omega \subseteq \mathbb{C}$ of eigenvalues determines a holomorphic Hermitian vector bundle E_T. One of the basic theorems they prove states that the unitary equivalence class of the operator T and the equivalence class of the holomorphic Hermitian vector bundle E_T are in one to one correspondence. This correspondence appears somewhat mysterious until one detects the invariants for the vector bundle E_T in the operator T and vice-versa. Fortunately, this is possible in some cases. Thus they point out that if the operator T possesses the additional property that the dimension of the eigenspace at ω is 1 for all $\omega \in \Omega$ then the map $\omega \mapsto \ker(T - \omega)$ admits a non-zero holomorphic section, say γ, and therefore defines a line bundle $\mathcal{L}_T$ on Ω. As is well known, the curvature $\mathcal{K}_{\mathcal{L}}$ defined by the formula

$$\partial\bar{\partial} \log \|\gamma(\omega)\| d\omega \wedge \bar{d}\omega$$

is a complete invariant for the line bundle $\mathcal{L}_T$. On the other hand, define

$$N_T(\omega) = (T - \omega)|_{\ker(T-\omega)^2}, \ \ \omega \in \Omega$$

and note that $N_T(\omega)^2 = 0$. It follows that if T is unitarily equivalent to $\tilde{T}$, then the corresponding operators $N_T(\omega)$ and $N_{\tilde{T}}(\omega)$ are unitarily equivalent for all $\omega \in \Omega$. However, Cowen and Douglas prove the non-trivial converse, namely that if $N_T(\omega)$ and $N_{\tilde{T}}(\omega)$ are unitarily equivalent for all $\omega \in \Omega$ then T and $\tilde{T}$ are unitarily equivalent. What does this have to do with the line bundles $\mathcal{L}_T$ and $\mathcal{L}_{\tilde{T}}$? To answer this question, we must ask what is a complete invariant for the unitary equivalence class of the operator $N_T(\omega)$. To find such a complete invariant we represent $N_T(\omega)$ with respect to the orthonormal basis obtained from the two linearly independent vectors $\gamma(\omega), \partial\gamma(\omega)$ by Gram-Schmidt orthonormalization process. Then an easy computation shows that

$$N_T(\omega) = \begin{pmatrix} 0 & \mathcal{K}_{\mathcal{L}}(\omega)^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \ \ \omega \in \Omega.$$

It then follows that $\mathcal{K}_{\mathcal{L}}(\omega)$ is a complete invariant for $N_T(\omega)$, $\omega \in \Omega$. This explains the relationship between the line bundle $\mathcal{L}_T$ and the operator T in an explicit manner.

Subsequently, in the paper "Operators Possesing an Open Set of Eigenvalues", Cowen and Douglas define a class of commuting operators possessing an open set of eigenvalues and attempt to provide similar computations as above. However, they give the details only for a pair of commuting operators. While the results of that paper remain true in the case of an arbitrary n - tuple of commuting operators, it requires additional effort which we explain in this thesis.